Supersymmetry in Stochastic Processes with Higher-Order Time Derivatives

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A supersymmetric path integral representation is developed for stochastic processes whose Langevin equation contains any number N of time derivatives, thus generalizing the Langevin equation with inertia studied by Kramers, where $N=2$. The supersymmetric action contains N fermion fields with first-order time derivatives whose path integral is evaluated for fermionless asymptotic states.

1. For a stochastic time-dependent variable x_t obeying a first-order Langevin equation

$$L_t[x] \equiv \dot{x}_t + F(x_t) = \eta_t, \tag{1}$$

driven by a white noise η_t with $\langle\eta_t\rangle = 0$, $\langle\eta_t\eta_{t'}\rangle = \delta_{tt'}$, the correlation functions $\langle x_{t_1}\cdots x_{t_n}\rangle$ can be derived from a generating functional

$$Z[J] = \langle e^{i\int dtJx}\rangle = \int \mathcal{D}x\,\Delta\, e^{-S_b+i\int dtJx}, \tag{2}$$

with an action $S_b = \frac{1}{2}\int dt\, L_t^2$, and a Jacobian $\Delta = \det\delta_{x_{t'}}L_t$. We denote by $\delta_{x_{t'}}L_t$ the functional derivative of $L_t[x]$ with respect to its argument. Explicitly: $\delta_{x_{t'}}L_t = [\partial_t + F'(x_t)]\delta_{tt'}$. The time variable is written as a subscript to have room for functional arguments after a symbol. It was pointed out by Parisi and Sourlas [1] that by expressing the Jacobian Δ as a path integral over Grassmann variables

$$\Delta = \int \mathcal{D}\bar{c}\mathcal{D}c\, e^{-S_f} \tag{3}$$

with a fermionic action

$$S_f = \int dtdt'\,\bar{c}_t\,\delta_{x_{t'}}L_t\,c_{t'} = \int dt\,\bar{c}_t(\partial_t + F')c_t\ , \tag{4}$$

the combined action $S \equiv S_b + S_f$ becomes invariant under supersymmetry transformations generated by the nilpotent ($Q^2=0$) operator

$$Q = \int dt\,(ic\delta_x - iL_t\delta_{\bar{c}}) \tag{5}$$

The supersymmetry implies $QS=0$.

The determinant (3) should not be confused with the partition function for fermions governed by the Hamiltonian associated with the action (4). Instead of a trace over external states it contains only the vacuum-to-vacuum transition amplitude for the imaginary-time interval under consideration. In the coherent state representation, $\bar{c}_t$ and c_t are set to zero at the initial and final times, respectively [2].

The path integral (2) can also be rewritten in a canonical Hamiltonian form by introducing an auxiliary Gaussian integral over momentum variables p_t, and replacing S_b by $S_b^H = \int dt\,(p_t^2/2 - ip_tL_t)$. The generator of supersymmetry for the canonical action is $Q^H = \int dt\,(ic_t\delta_{x_t} + p_t\delta_{\bar{c}_t})$. This form has an important advantage to be used later that it does not depend explicitly on D_t, so that the above analysis remains valid also for more general colored noises with an arbitrary correlation function $\langle\eta_{at}\eta_{bt'}\rangle = (D_{ab})_{tt'} \neq \delta_{tt'}$.

Inserting (3) into (2), the generating functional becomes

$$Z[J] = \langle e^{i\int dtJx}\rangle = \int \mathcal{D}p\mathcal{D}x\mathcal{D}\bar{c}\mathcal{D}c\ e^{-S^H - S_f + i\int dtJx}. \tag{6}$$

This representation makes the stochastic process (1) equivalent to by supersymmetric quantum mechanical system in imaginary time. In the supersymmetric formulation of a stochastic process, there exists an infinity of Ward identities between the correlation functions which can be collected in the functional relation

$$\int \mathcal{D}p\mathcal{D}x\mathcal{D}\bar{c}\mathcal{D}c\, e^{-S^H} Q^H\,\Phi = 0, \tag{7}$$

valid for an arbitrary functional $\Phi \equiv \Phi[p,x,\bar{c},c]$. The Ward identities simplify a perturbative computation of the correlation functions.

A proof of the equivalence of (1) to (2) requires a regularization of the path integral, most simply by time slicing. This is not unique, since there are many ways of discretizing the Langevin equation (1). If one sets $t_i = i\epsilon$, for $i = 0,1,2,\ldots,M$, $x_i = x_{t_i}$, and $F_i = F(x_i)$, then the velocity $\dot{x}$ may be approximated by $(x_i - x_{i-1})/\epsilon$. On the sliced time axis, the force $F(x_t)$ may act at any time within the slice (t_i, t_{i-1}), which is accounted for by a parameter a and a discretization $F \to aF_i + (1-a)F_{i-1}$. Note that the discretized Langevin equation is assumed to be causal, meaning that given the initial value of the stochastic variable x_0 and the noise configurations

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$\eta_0, \eta_1, ..., \eta_{M-1}$, the Langevin equation uniquely determines the configurations of the stochastic variable at later time, $x_1, x_2, ..., x_M$. The simplest choice of the right-hand side of the Langevin equation compatible with the causality is to set it equal to η_{i-1}. In general, one can replace η_{i-1} by $\sum_1^M A_{j-1,i-1}\eta_{i-1}$ with A being an orthogonal matrix, $A^T A = 1$. The latter is just an evidence of the symmetry of the stochastic process with the white noise with respect to orthogonal transformations $\eta_t \to (A\eta)_t$.

Some specific values of the interpretation parameter a have been favored in the literature, with $a = 0$ or $1/2$ corresponding to the so-called Itô- or Stratonovich-related interpretation of the stochastic process (1), respectively [2,3]. In the time-sliced path integral, these values correspond to a prepoint or midpoint sliced action [2,4]. Emphasizing the a-dependence of the sliced action, we shall denote it by S_a^H. This action is supersymmetric for any a: $Q^H S_a^H = 0$, and the sliced generator $Q^H = \sum_i (ic_i\partial_{x_i} + p_i\partial_{\bar{c}_i})$ turns out to be independent of both the interpretation parameter a and the width ϵ of time slicing [2]. A shift of a changes the action by the Q-exact term,

$$S_{a+\delta a}^H = S_a^H + \delta a\, Q^H G, \tag{8}$$

where G is a function of a and a functional of $p, x, \bar{c}, c$. This makes the Ward identities independent of a, i.e. on the interpretation of the Langevin equation. Indeed, setting $\delta a = -a$ we find $e^{-S_a^H} = e^{-S_0^H} e^{aQ^H\Phi} \equiv e^{-S_0^H}(1 + Q^H\Phi'_a)$. Substituting this relation into (7) we observe that the a-dependence drops out from the Ward identities because of the supersymmetry $Q^H S_0^H = 0$ and the nilpotency $(Q^H)^2 = 0$.

The simplest situation arises for the Itô choice, $a = 0$. Then the sliced fermion determinant Δ becomes a trivial constant independent of x. In the continuum limit of the path integral, however, this choice is inconvenient since then the limiting action S_0 *cannot* be treated as an ordinary time integral over the continuum Lagrangian. Instead, $S_{a=0}$ goes over into a so-called Itô stochastic integral [2]. The Itô integral calculus [3] differs in several respects form the ordinary one, most prominently by the property $\int dx \neq \int dt\dot{x}$. This difficulty is avoided taking the Stratonovich value $a = 1/2$, for which the continuum limit of $S_{1/2}$ *is* an ordinary integral [2,5]. Splitting (8) as $S_a = S_{1/2} + (a - 1/2)Q\, G$, the non-Stratonovich part vanishes in the continuum limit because Q does not depend on the slicing parameter ϵ, whereas G is proportional to $\epsilon \to 0$ [2]. For $a = 1/2$, formula (6) has a conventional continuous interpretation as a sum over paths, and can be treated by standard rules of path integration (e.g., perturbation expansion around Gaussian measures). The price for this is the additional fermion interaction, which possesses as an attractive feature the additional supersymmetry.

The aim of our work is to extend this supersymmetric path integral representation to stochastic processes with higher time derivatives

$$L_t = \gamma(\partial_t)\dot{x}_t + F(x_t) = \eta_t\ , \tag{9}$$

where γ is a polynomial of any order $N-1$, thus producing N time derivatives on x_t. This Langevin equation may account for inertia via a term $m\,\partial_t$ in $\gamma(\partial_t)$, and/or an arbitrary nonlocal friction $\int d\tau\gamma_\tau\dot{x}_{t-\tau} = \sum_{n=0}^{N-1}\gamma_n\partial_t^{+1}x_t$ where $\gamma_n = \int d\tau\gamma_\tau(-\tau)^n/n!$. The main problem is to find a proper representation of the more complicated determinant $\Delta = \det\delta_{x_{t'}}L_t$ in terms of Grassmann variables. The standard formula (3), though formally applicable, does not provide a proper representation of the determinant of an operator with higher-order derivatives because of the boundary condition problem. This problem is usually resolved via an operator representation of the associated fermionic system. In the stochastic context it has so far been discussed only for the single time derivative [2]. In the first-order formalism, the fermion path integral can be defined in terms of coherent states [2] with the above discussed vacuum-to-vacuum boundary conditions. Higher-derivative theories, however, have many unphysical features, in particular states with negative norms [6,7], and it is *a priori* unclear how to define the boundary conditions for the associated fermionic path integral. In gauge theories, the Faddeev-Popov ghosts give an example of a fermionic theory with higher-(second-)order derivatives. There, unphysical consequences of the negative norms of the ghost states are avoided by imposing the so called BRST invariant boundary conditions upon the path integral. For the above stochastic determinant with higher-order derivatives, the correct boundary condition are unknown.

2. The solution proposed by us in this work is best illustrated by first treating Kramers' process where one more time derivative is present, accounting for particle inertia, i.e. where $\gamma(\partial_t) = \partial_t + \gamma$ for a unit mass $m \equiv 1$. Omitting the time subscript of the stochastic variables, for brevity, we replace the stochastic differential equation (9) by two coupled first-order equations

$$L_v = \dot{v} + \gamma v + F(x) = \nu_v, \tag{10}$$
$$L_x = \dot{x} - v = \nu_x, \tag{11}$$

There are now two independent noise variables, which fluctuate according to the path integral

$$\langle F[x,v]\rangle = \int \mathcal{D}\nu_x\mathcal{D}\nu_v F[x,v] \times e^{-1/2\int dt[\nu_v^2/2(1-\sigma)+(\dot{\nu}_x+\gamma\nu_x)^2/2\sigma]}\ . \tag{12}$$

A parameter σ regulates the average size of deviations of $\dot{x}$ from v in Eq. (11). If we regard the basic noise correlation functions as functional matrices $(D_n)_{tt'} = \langle\nu_{nt}\nu_{nt'}\rangle$ for $n = x, v$, which act on functions of time as linear operators $D_n f_t = \int dt'\,(D_n)_{tt'}f_{t'}$, the noise correlation functions associated with (12) are

$$D_v = 1 - \sigma \ , \quad D_x = \sigma e^{-\gamma t} \left(-\partial_t^{-1} e^{2\gamma t} \partial_t^{-1} \right) e^{-\gamma t} \ . \quad (13)$$

Substituting (11) into (10) we find the two-derivative version of (9), $\ddot{x} + \gamma \dot{x} + F(x) = \eta_\sigma$, driven by the combined noise

$$\eta_\sigma = \nu_v + \dot{\nu}_x + \gamma \nu_x. \quad (14)$$

This noise is white for *any* choice of σ:

$$\langle \eta_{\sigma t} \rangle = 0 \ , \quad \langle \eta_{\sigma t} \eta_{\sigma t'} \rangle = \delta_{tt'} \ . \quad (15)$$

Let $x_t[\eta]$ be a solution of the original Langevin equation (9), and $x_t[\eta_\sigma]$ a solution of the system (11), (10). The property (15) implies that $x_t[\eta_\sigma]$ has the same correlation functions as $x_t[\eta]$, for *any* σ, thus describing the same stochastic process. The freedom in choosing σ will later be used to make the effective supersymmetric action local in time.

Once we have transformed Kramers' process into a system of coupled first-order Langevin equations (11) and (10) which is a trivial extension of the first-order equation (1) to a matrix form, there obviously exists a path integral representation analogous to (6). It is for this reason that we have introduced a two noise variable and a fluctuating relation between $\dot{x}$ and v in Eq. (11). There is a complication though in that the noise ν_x is no longer white since D_x is nonlocal in time. However, as observed above this does not affect the supersymmetry in the canonical form (6) of the path integral since the supersymmetry generator Q^H does not depend on D_x (in contrast to Q).

Thus, having established the supersymmetric path integral representation of the equivalent first-order stochastic system, our strategy is to integrate out all auxiliary variables we have introduced and, thereby, derive the proper boundary conditions for the fermionic path integral in the higher order stochastic processes as well as to construct the supersymmetry generator in the initial configuration space.

To prepare the notation for the later generalization to a stochastic differential equation with N derivatives, we rename the variables x and v as x_n, with $\alpha = 1, N$, and for the moment $N = 2$. Only the equation for x_N contains the force $F = F(x_1)$. The other equation just establishes a fluctuating equality between $\dot{x}_1$ and x_2, the original process being described by $x \equiv x_1$. Inserting the stochastic equations (10) and (11) into the exponent of (12), we repeat the previous procedure and, choosing midpoint slicing with $a = 1/2$ à la Stratonovich, we obtain the path integral representation of the generating functional

$$Z[J] = \int \mathcal{D}p \mathcal{D}x \mathcal{D}\bar{c} \mathcal{D}c \, e^{-S^H + i \int dt J x} \ , \quad (16)$$

$$S^H = \sum_{n=1}^{2} \int dt \left(\frac{1}{2} p_n D_n p_n - i p_n L_n + \bar{c}_n \delta_{x_m} L_n c_m \right) \ .$$

The generator of supersymmetry is

$$Q^H = \sum_{n=1}^{2} \int dt (i \bar{c}_n \delta_{x_n} - p_n \delta_{\bar{c}_n}) \ . \quad (17)$$

It is readily verified that $Q^H S^H = 0$, using the fact that $\sum_{kmn} \bar{c}_m c_k (\delta_{z_k} \delta_{z_n} L_m) c_n \sim \sum_n c_n^2 = 0$ due to the Grassmann nature of c_n. Explicitly, the Fermi part of the action S^H reads

$$S_f = \int dt \left[\bar{c}_x \dot{c}_x + \bar{c}_v \dot{c}_v - \bar{c}_x c_v + \bar{c}_v c_x F'(x) + \gamma \bar{c}_v c_v \right] . \quad (18)$$

The Gaussian path integral over momenta in (16) has a meaning without time slicing, and can be performed to recover the Lagrangian version of the supersymmetric action

$$S = \sum_{n=1}^{2} \int dt \ \frac{1}{2} L_n D_n^{-1} L_n + S_f \ . \quad (19)$$

The associated generator of supersymmetry is obtained from (17) by substituting into Q^H the solutions of the Hamilton equations of motion $p_n = i D_n^{-1} L_n$ which extremize S^H ($\delta_{p_n} S^H = 0$), leading to

$$\tilde{Q} = \sum_{n=1}^{2} \int dt (i c_n \delta_{x_n} - i D_n^{-1} L_n \delta_{\bar{c}_n}) \ . \quad (20)$$

The final step consists in integrating out the auxiliary variable $x_2 = v$, which only ap pears quadratically in the bosonic part of the action. Making use of the explicit form of D_n given in (15), we obtain the Lagrangian form of the supersymmetric action

$$S_\sigma = \int dt \ \frac{1}{2} L_t \left(1 + \frac{1}{1 - \sigma} \partial_t D_x \partial_t \right) L_t + S_f \ , \quad (21)$$

where L_t is now the left-hand side of the initial equation (9), for the Kramers process at hand: $L_t = \ddot{x}_t + \gamma \dot{x}_t + F(x_t)$. At this stage, the effective action is nonlocal in time. Now we take advantage of the freedom in choosing the parameter σ. We go to the limit $\sigma \to 0$, in which case $D_x \sim \sigma$ vanishes, reducing the action to the local form

$$S = S_0 = \int dt \frac{1}{2} L_t^2 + S_f \ . \quad (22)$$

To find the generator of supersymmetry in this representation, we omit $\delta_{x_N} \equiv \delta_v$ in (20), and replace $x_N \equiv v$ by the solution of the equation of motion

$$\delta_v \tilde{S} = -D_x^{-1} L_x + \frac{1}{1 - \sigma} (-\partial_t + \gamma) L_v = 0 \ . \quad (23)$$

In the limit $\sigma \to 0$, $D_x^{-1} \sim \sigma^{-1}$ diverges, leading to an exact equality $L_x = v - \dot{x} = 0$, rather than the fluctuating one (11). To take the limit $\sigma \to 0$ in the operator (20), one must first substitute (23) into the would-be singular term $D_x^{-1} L_x$ in $\tilde{Q}$, and then take the limit. The supersymmetry generator assumes the final form

$$Q = \int dt \, [ic_x \delta_x - i(-\partial_t + \gamma) L_t \delta_{\bar{c}_x} - i L_t \delta_{\bar{c}_v}] \ . \tag{24}$$

The action (22) provides us with the desired supersymmetric description of processes with second-order time derivatives. An important feature of the supersymmetry generated by Q is that the supermultiplet contains one boson field and two fermion fields. The reason for this is, of course, that a boson field with N time derivatives in the action carries N particles, each of which must have a supersymmetric fermionic partner. The fermion degrees of freedom have the conventional first order action, which permits us to impose the vacuum-to-vacuum boundary conditions within the coherent state representation of fermionic path integrals [2]. The boundary conditions for the bosonic path integral are the causal ones: $x_{t=0} = x_0$ and $\dot{x}_{t=0} = v_0 = \dot{x}_0$.

We have circumvented the problem of the boundary condition for the determinant of a higher-order operator by enlarging the number of Fermi fields, thereby reducing the problem to the known one for the determinant of the single-derivative operator. What happens if we integrate out the auxiliary Grassmann variables $\bar{c}_v, \ c_v$. In these variables, the action (18) is harmonic, driven by external forces $\bar{c}_x$ and $c_x F'(x)$. After a quadratic completion the integration with the vacuum-to-vacuum boundary condition yields $\det(\partial_t + \gamma)$. The effective action for the other fermion pair becomes non-local

$$S_f = \int dt \, \left[\bar{c}_x \dot{c}_x + \bar{c}_x (\partial_t + \gamma)^{-1} (F'(x) c_x) \right] \ . \tag{25}$$

The total effective action $S = S_b + S_f$ is still supersymmetric. The supersymmetry is generated by the operator (24), if the last term in Q is dropped. The action (25) is the first-order action. So with the vacuum-to-vacuum boundary condition the integral over $\bar{c}_x, c_x$ would also give a determinant. Thus we get the representation

$$\Delta = \det[\partial_t + \gamma] \det[\partial_t + (\partial + \gamma)^{-1} F'(x)] \ . \tag{26}$$

Invoking the formula for the determinant of a block matrix, the non-locality in the second determinant can be removed, while maintaining the linearity in the time derivative

$$\Delta = \det \begin{pmatrix} \partial_t + \gamma & F' \\ -1 & \partial_t \end{pmatrix}, \tag{27}$$

which is exactly the determinant arising from the two-noise process (10), (11). In this way we have represented the determinant of the second-order operator as a determinant of a first-order operator acting upon a higher-dimensional space for which the boundary conditions are known.

Thus, with the help of two coupled equations driven by auxiliary noises we have succeeded in giving a unique meaning to the path integral representation of the Kramers process. The final path integral can be time-sliced in any desired way (prepoint, postpoint, midpoint, or any combination of these)—as long as the slicing is done equally in the bosonic and the fermionic actions. In Section 4, the procedure will be generalized to a friction coefficient γ which is a function of x.

3. We now generalize our construction to stochastic processes of an arbitrary order N. As a result we shall arrive at a supersymmetric extension of general higher order Lagrangian systems with a supermultiplet of N fermion fields which all possess a good quantum theory due to their first-order dynamics.

Consider a system of coupled stochastic processes

$$L_N = \dot{x}_N + \sum_{n=1}^{N} \gamma_{n-1} x_n + F(x_1) = \nu_N \ ; \tag{28}$$

$$L_n = \dot{x}_n - x_{n+1} = \nu_n \ , \quad n = N-1, N-2, ..., 1 \ , \tag{29}$$

where $x_1 \equiv x$. This stochastic process is equivalent to the original one if we assume the noise average as being taken with the weight e^{-S_ν}, generalizing that in (12) to

$$S_\nu = \frac{1}{2} \int dt \left[\frac{1}{1-\sigma} \nu_N^2 + \sum_{n=1}^{N-1} \frac{1}{\sigma_n} (\Lambda_{N-n} \nu_n)^2 \right] \ , \tag{30}$$

where $\sigma = \sum_{n=1}^{N-1} \sigma_n$ and $\Lambda_n = \sum_{m=0}^{n} \gamma_{N-m} \partial_t^{n-m}$, $\gamma_N \equiv 1$. As for $N = 2$, equations (28) and (29) can b e combined into a single equation $L_t = \nu_{Nt} + \sum_{n=1}^{N-1} \Lambda_{N-n} \nu_{nt} \equiv \eta_{\sigma t}$. From (30) follows that $\langle \eta_{\sigma t} \rangle = 0$ and $\langle \eta_{\sigma t} \eta_{\sigma t'} \rangle = \delta_{tt'}$. Thus the correlation functions of the system (28) are the same as of the original one. Note also that the combined noise correlation functions do not depend on the parameters σ_n. We shall assign some specific values to the σ's to simplify the sequel formalism.

The Hamiltonian path integral for the stochastic system (28) and (29) has the form (16), where the label n runs now from 1 to N. With the same extension of the index sum, the operator Q^H in (17) generates supersymmetry. The noise correlation functions (13) are generalized to $D_N = 1 - \sum_{n=1}^{N-1} \sigma_n$ and $D_n = \sigma_n (\Lambda_{N-n}^\dagger \Lambda_{N-n})^{-1}$. After integrating out the momenta p_n we arrive at the action (19) with the extended sum, and the generator of supersymmetry assumes the form (20) with the extended sum.

Integrating out the auxiliary variables x_n is now technically more involved, but the integral is still Gaussian. A successive integration is possible by observing that the fermion action does not depend on the variables x_n for $n > 1$, the stochastic process being nonlinear only in the physical variable $x_1 \equiv x$. The classical equations of motion $\delta_{x_n} \tilde{S} = 0$ can be written in the form

$$-D_{n-1}^{-1} L_{n-1} - \partial_t D_n^{-1} L_n + \gamma_{n-1} D_N^{-1} L_N = 0 \ , \tag{31}$$

for $n = 2, 3, ..., N$. Combining the equations for $n = N$ and $n = N-1$, and the result with the equation for $n = N-2$, and so on, we derive the relation

$$D_n^{-1} L_n = \frac{1}{1-\sigma}\left[\sum_{k=0}^{N-n}(-1)^k \gamma_{n+k}\partial_t^k\right] L_N \ , \tag{32}$$

having inserted $D_N = 1-\sigma$ and with $n = 2, 3, ..., N$. These expressions may be substituted into the action (19), and the generator (20). As in the case $N = 2$, the supersymmetric Lagrangian action and the operator Q turn out to have a smooth limit $\sigma_n \to 0$. Since $D_n \sim 1/\sigma_n$, we see from (32) that $L_n \to 0$, and we recover the physical relations $\dot{x}_n = x_{n+1}$ and, hence, $x_n = \partial_t^n x$. The action assumes the form (22), with L_t of Eq. (9). The generator of supersymmetry becomes

$$Q = i\int dt \left\{ c_1\delta_x - \sum_{n=1}^{N}\left[\sum_{k=0}^{N-n}(-1)^k\gamma_{n+k}\partial_t^k L_t\right]\delta_{\bar{c}_n}\right\} \ . \tag{33}$$

For convenience, we give the fermion action explicitly:

$$S_f = \int dt \left[\sum_{n=1}^{N}\bar{c}_n\dot{c}_n - \bar{c}_n c_{n+1} + c_N\left(\sum_{n=0}^{N}\gamma_{n-1}c_n + F'(x)c_1\right)\right]. \tag{34}$$

The operator (33) transforms the original stochastic variable $x = x_1$ into the Grassmann variable c_1, $Qx = ic_1$, whereas all the fermionic variables are transformed into some functions of the only bosonic variable x. The fermionic action (34) is constructed in such a way that QS_f depends only on c_1. The terms containing the other Grassmann variables are cancelled amongst each other. The c_1-term is cancelled against the term resulting from QS_b, i.e. $Q(S_b + S_f) = 0$. It is important to realize that the fermions are coupled with each other, and thus belong to an irreducible supermultiplet. The number of fermion is equal to the highest order of the time derivative entering the bosonic action, as observed before for $N = 2$.

4. The idea of splitting the higher order Langevin equation into a system of coupled first-order stochastic processes with a combined noise can also be applied to construct a supersymmetric quantum theory associated with the higher order stochastic process where the coefficients γ_n are functions of x_t. We illustrate this with the example of Kramers' process with the friction coefficient being a function of the stochastic variable x_t.

A straightforward replacement of γ by $\gamma(x)$ in (10) would yield a problem because the combined noise η_σ appears to be a function of x_t, making the system (10), (11) inequivalent to the original stochastic process (if the Gaussian distributions for the auxiliary noises are assumed). To resolve this problem, we take two coupled non-linear first-order processes

$$L_v = \dot{v} + v + \lambda_v(x) = \nu_\sigma \ , \tag{35}$$

$$L_x = \dot{x} - v + \lambda_x(x) = \nu_\sigma \ . \tag{36}$$

The functions $\lambda_{x,v}$ are subject to the condition

$$\lambda_x' = \gamma - 1 \ , \quad \lambda_v = F - \lambda_x \ . \tag{37}$$

With the noise average defined by (12) and the condition (37), the stochastic system (35), (36) is equivalent to the original system $L_t = \ddot{x} + \gamma(x)\dot{x} + F(x) = \eta$.

The difference between (36) and (11) is just the extra force λ_x, which does not affect the derivation of the associated supersymmetric action. Repeating calculations of section 2, we arrive at the supersymmetric action $S = S_b + S_f$ where

$$S_b = \frac{1}{2}\int dt\,(\ddot{x} + \gamma(x)\dot{x} + F(x))^2 \ , \tag{38}$$

$$S_f = \int dt\,[\bar{c}_x\dot{c}_x + \bar{c}_v\dot{c}_v + \bar{c}_x c_x(\gamma(x) - 1) + \bar{c}_v c_v - \bar{c}_v c_x(F'(x) - \gamma(x) + 1) - \bar{c}_x c_v] \,. \tag{39}$$

The supersymmetry generator has the form

$$Q = \int dt\,(ic_x\delta_x - iL_t\delta_{\bar{c}_v} - i(-\partial_t + 1)L_t\delta_{\bar{c}_x}) \ . \tag{40}$$

It is not hard to verify that $QS = 0$.

If we set γ to be independent of x in (39), the fermionic action does not turn into (18), in contrast to what one might expect. The reason is that the fermionic path integral exhibits a large symmetry associated with general canonical transformations on the Grassmann phase space spanned by $\bar{c}$ and c. Recall that under canonical transformations the canonical one-form $\sum_n \bar{c}_n dc_n$ is invariant up to a total differential $dF(\bar{c}, c)$. Also the measure $\prod_n d\bar{c}_n dc_n$ remains unchanged. Thus there exists infinitely many equivalent supersymmetric representations of the same stochastic process. The situation is similar to the BRST symmetry [7] in gauge theories where the BRST charge is defined up to a general canonical transformation. This freedom can be used to simplify the fermionic action or the Fermi-part of the supersymmetry generator.

This formal invariance of the continuum phase-space path integral measure with respect to canonical transformations has been studied thoroughly [8] for bosonic phase spaces. A regularization of the continuum phase-space path integral measure with respect to canonical transformations on a phase space which is a Grassmann manifold is still an open problem.

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